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Resonant interactions between waves. The case of discrete oscillations

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The mathematical basis for resonance is investigated using a model equation describing one-dimensional dispersive waves interacting weakly through a quadratic term. If suitable time-invariant boundary conditions are imposed, possible oscillations of infinitesimal amplitude are restricted to a discrete set of wave-numbers. An asymptotic expansion valid for small amplitude shows that oscillations of different wave-number interact primarily in independent resonant trios. Energy is redistributed between members of a trio over a characteristic time inversely proportional to the amplitude of the oscillations in a periodic manner. The period depends on the initial conditions but is in general finite. Cubic interactions through resonant quartets are also discussed. The methods used are valid for a fairly wide class of equations describing weakly non-linear dispersive waves, but the expansion procedure used here fails for a continuous spectrum.

1. Introduction

Phillips (1960) drew attention to the phenomenon of resonance while discussing the role of small non-linear terms in the theory of ocean waves. As a necessary preliminary to the construction of a complete statistical theory for a continuous spectrum, he investigated the interactions of pairs of infinite sinusoidal wave-trains of small, but not infinitesimal, amplitude. Infinitesimal waves are described by a linear equation of which these wave-trains are solutions in the normal modes. The largest non-linear terms in the equations give rise to sinusoidal velocity and pressure fields with wave-number and frequency equal to the sum (or difference) of the wave-numbers and frequencies of the primary waves, and proportional to the product of their amplitudes. These fields may be interpreted as the response of the system of oscillators described by the linear part of the equations to the forcing terms due to the non-linear part. This response is, in general, constant and very small, but an apparent exception occurs if the frequency of the forced oscillation is equal to the frequency of infinitesimal free waves of the same wave-number. *Resonance* then occurs, and the forced wave may build up after a sufficient number of oscillations to be comparable in magnitude with the primary waves. At this stage the perturbation scheme used by Phillips breaks down.

Phillips also pointed out that the frequency relation for ocean waves does not admit of *quadratic* resonances; but if three primary waves interact (two of which may be identical), *cubic* terms in the equations may force resonant responses in a fourth mode. Longuet-Higgins (1962) made further calculations of the interaction coefficients, and suggested an experiment for verification of the theory.

Benney (1962) showed that if the amplitudes of the three primary waves are regarded as slowly varying functions of time, rather than as strict constants, the interactions can be described by much simplified equations apparently valid even when the amplitude in the fourth mode has become comparable with that in the other three. He obtained some integrals of these equations indicating that the process can be regarded as one of energy sharing between all four modes, the maximum energy in each being limited by the initial conditions. He also showed that, if the wave numbers of the three primary modes are not such that the resulting cubic terms can force an oscillation which resonates, the amplitude in the fourth mode remains an order of magnitude smaller.

It will be shown below (§ 7) that a complete solution to Benney's equations may be obtained. This is a periodic function of time, the cycle of energy transfer for resonant or nearly resonant modes repeating itself after a long time which depends on the initial amplitudes and phases, but is, in general, finite. According to this model there is thus no tendency towards equipartition of energy. This conclusion depends on equations in which non-linear terms of higher order than the cube of the slope of the sea surface are neglected.

Meanwhile Hasselmann (1962) constructed a full statistical theory based on the postulate that waves of different wave-number are statistically independent. Using a formal expansion procedure he computed to lowest order the rate of transfer of energy between different parts of the spectrum. This transfer is associated with resonances of the type discussed by Phillips. In an earlier version of this paper (Bretherton 1963) the author criticized the postulate of statistical independence on the ground that if true initially it will not persist after a substantial transfer of energy has occurred. Subsequent correspondence has revealed that the effect of this on the rate of energy transfer is uncertain and these criticisms are being withheld for later publication.

The discussions described above are all concerned with ocean waves. The algebra involved in calculating the interaction coefficients can be daunting, and the fact that the lowest order resonances can occur only through cubic interactions, although quadratic non-resonant interactions are also present, complicates the analysis. The phenomenon of resonance is, however, of potential importance for any system of weakly interacting waves in a dispersive medium. Several workers (e.g. Ball 1964) have investigated its possibility in other contexts.

To clarify some aspects of the mechanism whereby energy is transferred between different wave-numbers we will illustrate them by reference to a simple system possessing the appropriate properties. A suitable example is afforded by the partial differential equation

$$\frac{\partial^2\psi}{\partial t^2}+\frac{\partial^4\psi}{\partial x^4}+\frac{\partial^2\psi}{\partial x^2}+\psi=\psi^2. \tag{1}$$

This is in dimensionless form and has time derivatives of the second order only, as well as the simplest possible non-linear term. However, all the methods used here are applicable to a much wider class of equations in which the coefficients are independent of time and position variables, the linear part describes dispersive waves, and the non-linear part may be expanded in powers of the dependent variable(s) and its derivatives. One reasonable restriction is that energy should be conserved. In the case of equation (1) we have

$$\frac{\partial}{\partial t}\int_a^b \frac{1}{2}\left\{\left(\frac{\partial\psi}{\partial t}\right)^2+\psi^2-\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial^2\psi}{\partial x^2}\right)^2-\tfrac{2}{3}\psi^3\right\}dx=\left[\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial x\,\partial t}-\left(\frac{\partial\psi}{\partial x}+\frac{\partial^3\psi}{\partial x^3}\right)\frac{\partial\psi}{\partial t}\right]_a^b,$$

so that

$$E=\frac{1}{2}\left\{\left(\frac{\partial\psi}{\partial t}\right)^2+\psi^2-\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial^2\psi}{\partial x^2}\right)^2\right\}-\tfrac{1}{3}\psi^3 \tag{2}$$

may be interpreted as an energy density, and

$$S=\left(\frac{\partial\psi}{\partial x}+\frac{\partial^3\psi}{\partial x^3}\right)\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial x\,\partial t}$$

as an energy flux. A disadvantage of this model is that the formulae for orders of magnitude do not display the dimensions of the quantities involved. However, the physical manifestations of resonance are so diverse that no unified interpretation can be given to the magnitude of a typical wave or oscillation, which is the small parameter here. Unless otherwise specified, all wave-numbers and frequencies are assumed to be of order unity, but amplitudes are very small.

In this paper the object is to investigate what simplifications can be made associated with resonance when discussing non-linear interactions between oscillations or waves of different wave-numbers and of asymptotically small amplitude. The main concern here is with oscillations, for which there is a discrete set of possible wave-numbers. An expansion scheme is presented (§§ 4, 5) in which equations similar to those proposed by Benney (1962) describe the magnitudes and phases of the oscillations of lowest order. In §§ 6, 7, solutions to these equations are discussed. A different situation arises for the resonant interactions of waves with a continuous spectrum. It is then not possible to isolate a trio of interacting wave-numbers, but only bands about wave-numbers satisfying the resonance condition of width decreasing with the wave amplitude. The author hopes to discuss this case in a subsequent publication, using methods developed from those presented here.

2. Notation and definitions

When the magnitude of ψ is infinitesimal, the quadratic term on the right-hand side of equation (1) may be neglected and the equation becomes linear. It is then customary to describe the general solution as the superposition of a set of normal modes,

$$\psi=a(k)\exp\left[i\{kx+n(k)t\}\right]+a(-k)\exp\left[i\{-kx+n(-k)t\}\right], \tag{3}$$

where

$$n(k)=\mp\sqrt{(1-k^2+k^4)}\quad\text{if}\quad k>0,$$

and

$$n(-k)=-n(k). \tag{4}$$

This sign convention has the minor advantage that the reality of ψ is assured by the simple condition

$$a(k) = a^*(-k), \tag{5}$$

where the asterisk denotes the complex conjugate, while the normal mode of equation (3) may be either a wave of amplitude $a^+(k)$ travelling in the positive Ox direction if the negative sign is taken in the first of equations (4), or a wave $a^-(k)$ in the opposite direction if the positive sign is taken. This double-valued nature of $n(k)$ will not concern us greatly.

It will be necessary also to consider sinusoidal functions like

$$b(k,n)\exp\{i(kx+nt)\},$$

where n does not satisfy the dispersion relation (4). These represent forced oscillations of the linear system described by the left-hand side of equation (1), as opposed to the natural oscillations which are the normal modes. In any case the requirement that ψ be real demands that for every pair (k,n) there corresponds the pair $(-k,-n)$, with

$$b(k,n) = b^*(-k,-n),$$

and the mode will be described by either wave-number frequency pair indiscriminately. For every equation for $a(k)$ or $b(k,n)$ there is a complex conjugate equation. This is the corresponding expression for $a(-k)$ or $b(-k,-n)$.

If suitable reflecting boundary conditions (e.g. $\partial\psi/\partial x = \partial^3\psi/\partial x^3 = 0$) are imposed at given points, e.g. $x = \pm X_0$, the function $\psi(x,t)$ defined between these points may be extended outside this range as a periodic function of x of period $4X_0$. Such a periodic function would remain periodic for all time to any order in the amplitude. The possible normal modes are then restricted to a discrete set of values of k and we will call them *oscillations*. When no boundary conditions are given k may have any real value and the normal modes are *waves*. Of course, in any given problem, if the boundaries are sufficiently distant it is appropriate to consider the discrete spectrum to be continuous, and the oscillations become waves. In the sequel we will estimate how distant the boundaries must be if this is to be true. If X_0 is large, the permitted lines in the possible spectrum are separated by a small but finite *line-spacing*

$$\delta k_0 = \pi/2x_0. \tag{6}$$

Permitted values of the frequency are also restricted, the *frequency-spacing* being approximately

$$\delta n_0 = c(k)\,\delta k_0,$$

where

$$c(k) = dn/dk$$

is the *group velocity*.

When ψ is small but not infinitesimal, it is still sensible to regard it as the superposition of expressions like that of equation (3), with the same frequency relation (4), but the complex amplitude must now be a slowly varying function of time, $a(k,t)$. If we define the *interaction time*, $\hat{t}$, as the time in which a typical amplitude changes by a significant fraction of itself,

$$\frac{1}{a}\frac{d}{dt}a(k,t) \sim \hat{t}^{-1}, \tag{7}$$

then, if $n(k)\,\hat{t} \gg 1$ for all relevant k, $a(k,t)$ is slowly varying and the interaction is *weak*. If $\hat{t} \sim 1$ or $\ll 1$, as in the theory of homogeneous turbulence, then the interaction is strong, we cannot regard the amplitudes as slowly varying, and none of the considerations of this paper apply. The reciprocal of the dimensionless interaction time, $\epsilon = \hat{t}^{-1}$, is the small parameter of this paper.

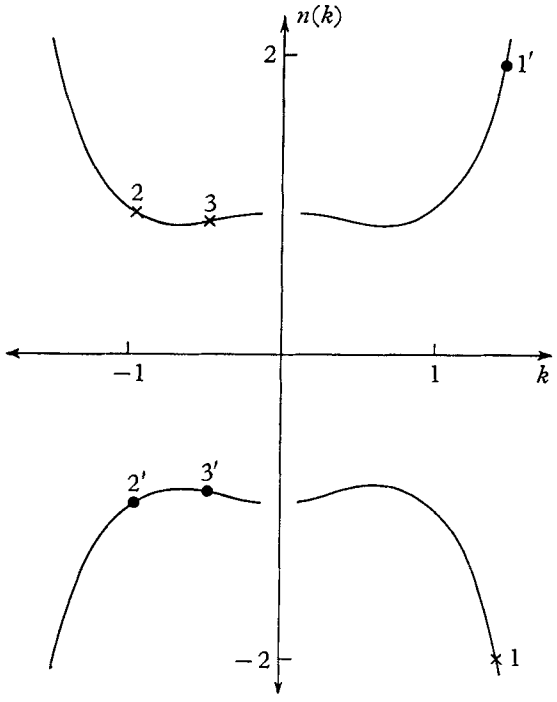

FIGURE 1. Two possible resonant trios if $n^2(k) = 1 - k^2 + k^4$. ×, Waves travelling in positive Ox direction. ●, Waves travelling in negative Ox direction.

The product of ψ for the two normal modes $(k_2, n_2(k_2))$, $(k_3, n_3(k_3))$ may be written in terms of sum and difference phases as

$$\begin{aligned} a_2^* a_3^* \exp[-i\{(k_2+k_3)x+(n_2+n_3)t\}] + a_{-2}^* a_{-3}^* \exp[i\{(k_2+k_3)x+(n_2+n_3)t\}] \\ + a_2^* a_{-3}^* \exp[-i\{(k_2-k_3)x+(n_2-n_3)t\}] \\ + a_{-2}^* a_3^* \exp[i\{(k_2-k_3)x+(n_2-n_3)t\}] \quad (8) \end{aligned}$$

Here we have used the shorthand notation a_{-2}^* for $a^*(-k_2)$, which is of course equal to a_2. Through the first two terms above, the quadratic term on the right-hand side of equation (1) thus acts as a forcing function for the linear oscillator in mode (k_2+k_3, n_2+n_3) described by the left-hand side, and through the third and fourth terms it forces in the mode (k_2-k_3, n_2-n_3). This effect must be summed over all pairs $\pm k_2$, $\pm k_3$ in the wave-number spectrum. However, Phillips (1960) pointed out that the response of the oscillators is very small, unless the forcing frequency happens nearly to coincide with the natural

frequency appropriate to that wave-number. Such a resonance occurs for wave-numbers in the neighbourhood of k_1, k_2, k_3, where

$$\left.\begin{aligned} k_1+k_2+k_3 &= 0,\\ n_1+n_2+n_3 &= 0,\end{aligned}\right\} \tag{9}$$

and

$$n_1 = n(k_1)$$

is a solution of the frequency relation (4). Equation (9) includes the possibility of resonance through difference phase if it is interpreted as needing to be satisfied only by either of the alternative ways (e.g. $(k_2, n_2), (-k_2, -n_2)$) describing each of the modes 1, 2 and 3.

When the condition (9) is written in this form it is obvious that if modes 2 and 3 can combine to resonate with mode 1, then modes 3 and 1 can resonate with 2 and modes 1 and 2 with 3. Thus if the change in amplitude in mode 3 is influenced by the amplitudes in modes 1 and 2, both the inverse processes will occur, and the three modes must be considered together as a resonant trio.

Suitable wave-numbers for a resonant trio are shown in figure 1. If k_1 is altered slightly, both k_2 and k_3 must be altered by determinate amounts to maintain the resonance conditions. There are thus two singly infinite families of possible trios for the frequency relation considered here, one family for waves travelling in the positive direction, the other similar family for waves in the negative direction. Within a certain band of wave-numbers, given any particular mode there is a unique pair of modes with which it resonates. We will confine our attention primarily to wave-numbers in this band.

3. Heuristic equations for resonant trio

In this section we will follow a heuristic approach, along the lines suggested by Benney (1962), investigating the interactions of a discrete set of oscillations in normal modes. It will be shown in §§ 4, 5 that this type of approach may be justified formally under certain conditions for oscillations, but not for waves, in the sense in which these terms are defined in § 2. A detailed study of this case is of value for comparison and contrast with the effects of resonance on groups of waves and within a continuous spectrum, which the author hopes to describe in a later paper.

Following Benney, we take in this section as our representation of a solution to equation (1),

$$\psi = \sum_k a(k,t) \exp\left[i\{kx + n(k)t\}\right]. \tag{10}$$

This differs from equation (3) in that the amplitude $a(k,t)$ in any mode is regarded as constant over a few oscillations but not over a time of order $\hat{t}$. The reality condition of equation (5) is still applicable, so terms are present in pairs $(k, n(k))$ and $(-k, -n(k))$. Equations will also occur in pairs, that for $a(-k,t)$ being the complex conjugate of that for $a(k,t)$.

If we substitute the representation (10) into the governing equation (1), the left-hand side becomes

$$\sum_k \left[\frac{d^2a}{dt^2} + 2in\frac{da}{dt} + \{-n^2 + (1-k^2+k^4)\}a\right] \exp\{i(kx+nt)\}.$$

The coefficient of $a(k)$ must vanish, because the frequency n has been chosen to satisfy the dispersion relation equation (4) for the linearized problem. The right-hand side of equation (1) consists of a sum of terms like expression (8). If now we divide by $\exp[i\{k_1x+n_1t\}]$, for some particular $(k_1, n_1(k_1))$, every term still contains a rapidly oscillating phase factor like $\exp i\{(k_2-k_1)x+(n_2-n_1)t\}$, with the exception of

$$\frac{d^2a_1}{dt^2}+2in_1\frac{da_1}{dt}$$

on the left-hand side and

$$2a_2^*a_3^*$$

on the right-hand side, where k_2, k_3 are uniquely related to k_1 by the condition that the relative phase

$$-(k_1+k_2+k_3)x-(n_1+n_2+n_3)t$$

should vanish identically, i.e. through the resonance condition (9). The factor 2 appears because the pair 2, 3 may arise in either order. If now we average over a time which is long compared to a typical period, but short compared to the interaction time $\hat{t}$, the contribution from all the rapidly oscillating terms is small and we are left with

$$\frac{d^2a_1}{dt^2}+2in_1\frac{da_1}{dt}=2a_2^*a_3^*$$

It is now seen that the assumption that $a_1(t)$ is slowly varying is only consistent if the magnitude of a_2a_3 is very much less than a_1, or in general if

$$|a| \ll 1 \quad \text{for all } k.$$

However, in this case the term d^2a_1/dt^2 is small compared to $n_1(da_1/dt)$ and may be neglected. There are similar expressions for a_2 and a_3, which wave-numbers also resonate with k_1, so finally the approximate equations are

$$\left.\begin{aligned} in_1(da_1/dt) &= a_2^*a_3^*,\\ in_2(da_2/dt) &= a_3^*a_1^*,\\ in_3(da_3/dt) &= a_1^*a_2^*. \end{aligned}\right\} \tag{11}$$

These describe the interactions of any trio which satisfies the resonance condition,

$$\begin{aligned} k_1+k_2+k_3 &= 0,\\ n_1+n_2+n_3 &= 0. \end{aligned}$$

Each trio thus behaves (to a first approximation at least) independently of all the others, and interactions take place within each resonant trio on a time scale $\hat{t}$ of order a^{-1}.

Weak points in this argument are that the precise meaning of slowly varying and rapidly oscillating functions is unspecified, except that the average value of the latter over a suitable time interval must be small, and also it is not clear what the next approximation to equations (11) should be or even how to set about attaining it. For oscillations, for which k, n are restricted to discrete values, there may be no values at all which satisfy the resonance condition exactly. How much tolerance is appropriate also needs clarification.

4. An explanation of the expansion procedure for oscillations

These difficulties may be resolved for oscillations, though not so easily for waves, by adopting a formal asymptotic approach based on two time variables. The author first started thinking along these lines following a suggestion from Prof. Benney himself. The precise arrangement necessary has caused considerable effort. It is based on the two time method due to Krylov and Bogoliubov (Minorsky 1962) but with inspiration from the matched expansions for low Reynolds number flow round spheres described by Proudman & Pearson (1957).

An unusual element is that although everything is arranged as an expansion

$$\psi(x,t) = \epsilon\, {}_1\psi + \epsilon^2\, {}_2\psi + \epsilon^3\, {}_3\psi + \ldots, \tag{12}$$

in successive powers of ϵ, the *nominal* order of magnitude so defined should be regarded only as a device for denumerating all the terms which arise when successive approximations

$$\epsilon\, {}_1\psi(x,t);$$
$$\epsilon\, {}_1\psi(x,t) + \epsilon^2\, {}_2\psi(x,t);$$
$$\epsilon\, {}_1\psi(x,t) + \epsilon^2\, {}_2\psi(x,t) + \epsilon^3\, {}_3\psi(x,t);\ \text{etc.}$$

are substituted into equation (1) and into the initial conditions

$$\psi(x,0) = F(x), \quad \partial\psi/\partial t(x,0) = G(x).$$

This procedure ensures that every term arising is included sequentially somewhere, but the crucial test of an asymptotic expansion is whether the remainders

$${}_1R(x,t); \quad {}_2R(x,t); \quad {}_3R(x,t), \text{ etc.},$$

after substitution of successive approximations have bounds which are of increasing order in ϵ uniformly in some domain. In this case, if $F(x)$ and $G(x)$ have continuous derivatives of all orders, and if

$$-X_0 \leqslant x \leqslant X_0, \quad 0 \leqslant \epsilon t \leqslant T_0,$$

for any T_0, we have

$$|{}_1R(x,t)| \leqslant {}_1B\epsilon^2; \quad |{}_2R| \leqslant {}_2B\epsilon^{\frac{5}{2}}; \quad |{}_3R| \leqslant {}_3B\epsilon^3; \text{ etc.}, \tag{13}$$

so the remainders increase in order of magnitude by only $\epsilon^{\frac{1}{2}}$ at each stage. Thus, to ensure that all the terms of *least possible order* ϵ^3 have been included, it is necessary to carry the prescription for constructing successive approximations to nominal order ϵ^5 (table 1 on p. 472). For the majority of terms the least possible order is equal to the nominal order, but it is not possible to say this for all. Thus equation (12) cannot be interpreted as meaning that there exists a function ${}_2\psi(x,t)$ which is independent of ϵ and has magnitude about 1, for it will be seen that ϵ enters explicitly into the prescription for ${}_2\psi$ and some terms entering ${}_2\psi$ have actual magnitude $\epsilon^{-\frac{1}{2}}$.

An example of this type of behaviour by the remainders is provided by the function

$$f(z,\epsilon) = \frac{2z}{\epsilon + 2z + z^2} \quad (z, \epsilon \geqslant 0);$$

which may be expanded either as

$$f(z,\epsilon) = \frac{1}{1+\frac{1}{2}z}\left\{1-\frac{\epsilon}{z(2+z)}+\frac{\epsilon^2}{z^2(2+z)^2}-\ldots\right\}$$

or, if $z = \epsilon\zeta$, as

$$f(z,\epsilon) \equiv \frac{2\zeta}{1+2\zeta+\epsilon\zeta^2} = \frac{2\zeta}{1+2\zeta}\left\{1-\epsilon\frac{\zeta^2}{1+2\zeta}+\epsilon^2\frac{\zeta^4}{(1+2\zeta)^2}-\ldots\right\}.$$

The first expansion is valid nominally when z is of order unity, but it may be used provided $\epsilon/z \to 0$ with ϵ. The second expansion is designed for $\zeta = z/\epsilon$ of order unity, but extends to any region in which $\epsilon\zeta = z \to 0$. The expansions are thus valid in overlapping regions and may be matched where $z \to 0$, $z/\epsilon \to \infty$. In this region the two approximations of nominal order zero are respectively

$$\frac{1}{1+\frac{1}{2}z} = 1-\tfrac{1}{2}z+\ldots,$$

and

$$\frac{2\zeta}{1+2\zeta} = 1-\frac{1}{2\zeta}+\ldots.$$

Thus a correct approximation in this region is $f(z) = 1$; but if we wish to describe $f(z,\epsilon)$ over the whole region $0 \leqslant z < \infty$, the minimum error is made by taking the second formula for $z \leqslant O(\epsilon^{\frac{1}{2}})$, and the first for $z \geqslant O(\epsilon^{\frac{1}{2}})$. The error is then bounded uniformly over the whole range, but is $O(\epsilon^{\frac{1}{2}})$ not $O(\epsilon)$. After a further approximation

$$f(z) = 1-\tfrac{1}{2}z-(\epsilon/2z) \quad \text{if} \quad z > \epsilon^{\frac{1}{2}},$$

or

$$f(z) = 1-(1/2\zeta)-\tfrac{1}{2}\epsilon\zeta \quad \text{if} \quad z \leqslant \epsilon^{\frac{1}{2}},$$

the largest neglected terms are

$$\tfrac{1}{4}\{z^2+O(z^3)\}+\tfrac{1}{2}\epsilon\{1+O(z)\}$$

or

$$\tfrac{1}{4}\{(1/\zeta^2)+O(1/\zeta^3)\}+\tfrac{1}{2}\epsilon\{1+O(1/\zeta)\},$$

which are both $O(\epsilon)$ at the matching point. It does not appear possible to find an expansion for $f(z,\epsilon)$ in powers of ϵ for which the order of the remainder increases by the same power at each stage uniformly over the whole range of z. In this paper, the analogue of the variable z is $n-n(k)$, the difference of the frequency n of an oscillation from the value $n(k)$ for natural oscillations of the same wave-number.

If ϵ is a small parameter describing the overall magnitude of the oscillations, we define two new time variables

$$T = \epsilon t, \quad \tau = t.$$

These are clearly not independent, once ϵ is specified, but they give rise to a different asymptotic structure as $\epsilon \to 0$. Almost all the solutions of equation (1) of small amplitude are roughly oscillatory, and we will describe the *phase* of an oscillation primarily in terms of the unscaled variable τ, but its complex *amplitude* in terms of the scaled variable T. This provides the formal codification of the idea of the amplitude as a 'slowly varying' function.

The first term of the expansion is of the form

$$ {}_1\psi(x,t) = \sum_k {}_1a(k,T)\exp[i\{kx+n(k)\tau\}]. \tag{14}$$

${}_1a(k,T)$ is an amplitude function which may vary by a substantial fraction over a scaled time $T = T_0$, but it is bounded and has least possible order unity, the same as its nominal order. ${}_2\psi(x,t)$ contains terms like ${}_2b(k,n,T)\exp[i\{kx+n\tau\}]$, describing forced oscillations which, in regions of wave-number-frequency space away from the curves $n = n(k)$, have amplitudes of order ϵ^2. There are also natural oscillations in the normal modes of nominal order ϵ^2,

$$ {}_2a(k,T)\exp[i\{kx+n(k)\tau\}].$$

Higher-order terms ${}_3\psi$, etc., are made up in a similar way.

The distinction between these two classes of forced and natural oscillations is not unambiguous, for if $n-n(k) = \epsilon N$, a quantity of order ϵ, an oscillation of frequency n may be described either as ${}_2b(k,n)\exp[i\{kx+n\tau\}]$ with ${}_2b(k,n)$ constant, or as ${}_1a(k,T)\exp[i\{kx+n(k)\tau\}]$ with ${}_1a(k,T)$ varying exponentially with phase NT. A similar confusion exists between ${}_2a$ and ${}_3b$, etc., which appear as terms of different order in the expansion (12).

We here settle the point by choosing a fixed positive M_0, and saying once and for all that if

$$ |n-n(k)| \leqslant \epsilon^{\frac{1}{2}}M_0 \tag{15}$$

the oscillation is described by ${}_1a(k,T)$, otherwise by ${}_2b(k,n,T)$; and similarly for higher-order oscillations. The mathematically natural description as forced oscillations appears when $n-n(k)$ is regarded as fixed as $\epsilon \to 0$; while the most interesting properties of natural oscillations appear for constant $N = \{n-n(k)\}/\epsilon$. The division (15) appears in a region where the two representations may be matched, and whereas for formal clarity it is essential to have in mind a given value for M_0, the ultimate results do not depend significantly upon it.

It will appear that for the amplitude ${}_1a(k_1)$ of one of the primary oscillations described by equation (14) to vary significantly with T, it must be at least partly resonant with two other modes 2 and 3. In other words, there must exist $k_2, n_2 = n(k_2)$ and $k_3, n_3 = n(k_3)$ such that

$$ \left.\begin{aligned} k_1+k_2+k_3 &= 0, \\ n_1+n_2+n_3 &= \epsilon N, \end{aligned}\right\} \tag{16}$$

where $N = O(1)$. For exact resonance, $N = 0$. But the expansion procedure described here is only applicable to *oscillations*, for which k and n may only assume certain discrete values.

If for given k_1, k_2 and k_3 are restricted by the first of equations (16), then as k_2 increases in steps of δk_0, k_3 decreases by similar amounts. The sum ϵN increases in steps approximately equal to $(c_2-c_3)\delta k_0$ where

$$ c_2 = dn/dk|_{k_2}$$

is the group velocity at k_2 and c_3 is similarly defined. Thus for given k_1 and $n_1(k_1)$ there will be, if ϵ is small enough, at most one permitted wave-number pair k_2, k_3

satisfying equation (16) with $|N|$ less than any predetermined value N_0. If the group velocities c_2 and c_3 are unequal, and if

$$2\epsilon N_0/|c_2-c_3|\,\delta k_0 < 1, \tag{17}$$

then at most one pair of modes can resonate with any particular mode 1.

This is the vital restriction imposed by the requirement of oscillations. For waves, δk_0 vanishes, and an infinity of pairs of wave-numbers take part simultaneously in partially resonant interactions with any given mode. The interaction equations are then much more complicated, and the whole order of magnitude structure of the asymptotic expansion is altered.

The condition (16) is in fact so stringent that for most modes 1 there will be no permitted modes at all with which significant quadratic resonant interactions will take place. For given k_1, and k_2 and k_3 restricted by the first of equations (16), possible values for ϵN are separated by intervals $|c_2-c_3|\,\delta k_0$, so the probability that for conditions chosen at random $|N|$ should be less than a predetermined value N_0 for any of these possibilities at all is

$$2\epsilon N_0/|c_2-c_3|\,\delta k_0.$$

Thus as ϵ tends to zero, the number of modes in which the primary oscillation undergoes any significant change with T will normally decrease to near vanishing point for most choices of $n(k)$ and δk_0. If the wave-numbers k of possible normal mode oscillations are fixed exactly, and independently of ϵ, only in the event that $N = 0$ for some trio will resonance persist for sufficiently small ϵ. In all other cases the trio becomes non-resonant in the limit $\epsilon \to 0$. The distinction between exactly resonant and completely non-resonant trios provides an example of a non-uniform limiting process. For some trios partial resonance will persist much longer than for others as ϵ becomes small.

To bridge this gap between non-resonant ($N = \infty$) and exactly resonant ($N = 0$) trios we retain the fictional possibility of the resonance condition being satisfied with error ϵN, where N is constant, finite, but not necessarily zero. Primary oscillations in such partially resonant modes are described in this scheme as natural oscillations by the functions ${}_1a(k, T)$. When the general solution for these is given in § 6, it will be seen that as $N \to \infty$, ${}_1a(k, T)$ approximates to a constant plus a sinusoidal perturbation of magnitude proportional to $1/N$ and phase NT (equation (28)). This perturbation is an approximate description of a forced oscillation ${}_2b(k, n)$ of nominally smaller order of magnitude, but amplitude approximately proportional to $(n-n(k))^{-1}$.

The fact that it is not possible to regard an oscillation in any *given* mode as partially resonant for arbitrarily small ϵ puzzled the author for a while. However, it is not necessary for it to be described by any particular term in our equations for every value of ϵ, provided that for any given ϵ we can arrive unambiguously at a sequence of approximations to $\psi(x, t)$ which contains somewhere contributions from every oscillation. For certain ϵ this may be as part of ${}_1a(k, T)$, but for smaller ϵ as ${}_2b(k, N)$. The match between these types of term mentioned in the previous paragraph shows that the conclusions should be independent of the precise choice of the transition point M_0, and indicates that

the expansions based nominally on $\{n-n(k)\}/\epsilon$ constant and on $n-n(k)$ constant do indeed have an overlapping range of validity. However, the error made by truncating the expansion procedure at any particular stage may have least possible order for the whole range of permitted n substantially lower than the nominal order for the stage.

The procedure for determining successive terms in the asymptotic expansion follows that for the standard two time method. The amplitude functions $\epsilon\,{}_1a(k,T)$ of the primary oscillations are left initially undetermined, except in order of magnitude. The amplitudes $\epsilon^2\,{}_2b(k,n,T)$ of the second order forced oscillations are expressed in terms of them. These amplitudes are nominally $O(\epsilon^2)$, but for $n-n(k)$ just outside the range of equation (15) the true value is $O(\epsilon^{-\frac{3}{2}})$. Then $d\{{}_1a(k,T)\}/dT$ is chosen so that the remainder of the forcing function on the right-hand side of the equation for ${}_2\psi$ vanishes, i.e. that part which has not contributed to the particular integrals ${}_2b$. If this were not done the solution for ${}_2\psi$ would necessarily be singular for $t=\tau\sim\epsilon^{-1}$. This was found by Phillips (1960), but the inclusion here of $d\,{}_1a/dT$ in the forcing function enables the validity of the expansion to be extended, but only if ${}_1a(k,T)$ varies slowly in a determinate way.

The equations describing variations of ψ up to and including terms of nominal order ϵ^2 also admit of the inclusion in ${}_2\psi$ of natural oscillations described by ${}_2a(k,T)$. The variation of these with T is at this stage undetermined, but for ${}_3\psi$ not to be singular when $t\sim\epsilon^{-1}$ it must be chosen in a certain way. General solutions of the equations for ${}_2a(T)$ are not available, but it may be seen that under certain circumstances $\epsilon^2\,{}_2a(k,T)$ may grow systematically with T until when $T\sim\epsilon^{-1}$ and $t\sim\epsilon^{-2}$ it is comparable with $\epsilon\,{}_1a(k,T)$. By this stage, the scheme presented here has broken down. This happens, for example, if no quadratically resonant trios are possible at all (not even partially resonant ones) but cubic resonances with four interacting modes may occur. Ocean waves provide an example of this. In terms of the scaled variable T all the amplitudes of the primary oscillations are constants, but ${}_2a(k,T)$ may grow linearly until after time $T\sim\epsilon^{-1}\,(t\sim\epsilon^{-2})$, it is comparable with ${}_1a(k)$. To describe this case the obvious procedure is to redefine T as ϵ^2t. In this paper, however, the interest is in quadratic resonances, and for any finite T_0 the estimates for the least possible orders of magnitude of ${}_2a$, etc., and the remainders ${}_2R$, etc., are valid in

$$0\leqslant t\leqslant\epsilon^{-1}T_0,$$

and a suitable sequence of these estimates may be found for any domain $0\leqslant t\leqslant\epsilon^{-(1+\alpha)}$, where $\alpha<1$. During either of these periods, arbitrarily large changes associated with quadratic resonances have had time to take place. This is in contrast to the procedure used by Phillips (1960), which was not valid uniformly in *any* domain $0\leqslant t\leqslant\epsilon^{-1}T_0$.

Once the expansion procedure has been carried successfully as far as ${}_2\psi$, there is no reason apart from algebraic complexity why the estimate for ψ given by equation (12) should not be refined further to include terms of any order. The derivative $d\{{}_3a(k,T)\}/dT$, for example, is the sum of an infinite number of products of terms of smaller nominal order. The number of these, however, arising from primary oscillations in any given wave-number band is finite, because of

the definite line and frequency spacing. For any reasonable distribution of amplitude $|_1a(k)|$ in the high wave-number region, the derivative should be well defined, and its magnitude should be comparable with that of the largest individual terms comprising it. When integrated over a definite interval $T \leqslant T_0$, this magnitude is also that of $_3a(k, T)$. The reasonable behaviour of $|_1a(k)|$ as $|k| \to \infty$, may be assured by assuming that the initial values $F(x), G(x)$ of ψ and $\partial\psi/\partial t$ are differentiable infinitely often when regarded as periodic functions of x of period $4X_0$.

Two further restrictions should be noted. The frequency relation $n(k)$ chosen here does not vanish for any real k. Resonances into a normal mode of zero frequency would require special consideration, as would real values of k at which there are branch points for $n(k)$. Also the group velocities c_1, c_2, c_3 for three partially resonant modes must be definitely unequal, otherwise there will be interactions between different trios in this neighbourhood, as well as within them. With these restrictions, the methods used here apply to any weakly non-linear system of dispersive waves with at least a quadratic non-linear term. The interaction coefficients entering the various equations will be different functions of k and n, but the structure will be unaltered. Finally, it cannot be too strongly emphasized that this picture depends crucially on

$$\delta k_0 > 0,$$

and cannot be used to describe a continuous spectrum of waves.

5. The formal expansion for oscillations

After these descriptions we proceed to the formal analysis. We have

$$\left\{\frac{\partial^2}{\partial t^2}+\frac{\partial^4}{\partial x^4}+\frac{\partial^2}{\partial x^2}+1\right\}\psi = \psi^2, \tag{1}$$

and
$$\psi = \epsilon\,{}_1\psi(x,t)+\epsilon^2\,{}_2\psi(x,t)+\epsilon^3\,{}_3\psi(x,t)+\ldots, \tag{12}$$

where
$${}_1\psi(x,t) = \sum_k a(k,T)\exp\left[i\{kx+n\tau\}\right] \tag{14}$$

with
$$T = \epsilon t, \quad \tau = t.$$

Now
$$\frac{d^2}{dt^2}\{\epsilon\,{}_1\psi\} = \sum_k\left\{\epsilon^3\frac{d^2\,{}_1a}{dT^2}+\epsilon^2\,2in\frac{d\,{}_1a}{dT}-\epsilon^2n^2\,{}_1a\right\}\exp\left[i\{kx+n\tau\}\right],$$

so on substitution into equation (1) and on comparison of terms of nominal order ϵ
$$\sum_k\{-n^2+(1-k^2+k^4)\}\,{}_1a(k,T)\exp\left[i\{kx+n\tau\}\right] = 0.$$

Thus from terms of order ϵ, we have that $_1a(k, T)$ vanishes unless

$$n^2 = n^2(k) = 1-k^2+k^4.$$

This is the conclusion of conventional linear theory, and gives no information about interactions.

The terms of order ϵ^2, on the other hand, yield

$$\left\{\frac{\partial^2}{\partial\tau^2}+\frac{\partial^4}{\partial x^4}+\frac{\partial^2}{\partial x^2}+1\right\}{}_2\psi = \sum_k -2in(k)\left\{\frac{d}{dT}\,{}_1a(k,T)\right\}\exp\left[i\{kx+n(k)\tau\}\right]$$
$$+\sum_{k_2,\,k_3} 2\,{}_1a^*(k_2,T)\,{}_1a^*(k_3,T)\exp\left[-i\{(k_2+k_3)x+(n_2+n_3)\tau\}\right]. \tag{18}$$

Equation (18) is a linear equation for the x, τ dependence of ${}_2\psi$. The partial derivatives of ${}_2\psi$ with respect to T enter first to nominal order ϵ^3, appearing as a forcing term on the right-hand side of the equation for ${}_3\psi$. When writing down the formal solution to equation (18) the derivatives $d\{{}_1a(k)\}/dT$, and the quadratic coefficients ${}_1a_2^*(T)\,{}_1a_3^*(T)$ on the right-hand side may be regarded as constants. A particular integral arising from the second term is

$$\sum_{k,n} {}_2b(k,n,T)\exp[i\{kx+n\tau\}],$$

where
$${}_2b(-k_2-k_3,-n_2-n_3,T) = \frac{2\,{}_1a_2^*(T)\,{}_1a_3^*(T)}{-(n_2+n_3)^2+\{n(k_2+k_3)\}^2},$$

or in its complex conjugate form,

$${}_2b(k_2+k_3,n_2+n_3,T) = \frac{2\,{}_1a_2(T)\,{}_1a_3(T)}{-(n_2+n_3)^2+\{n(k_2+k_3)\}^2}. \tag{19}$$

This is of bounded magnitude for all finite T, unless the denominator vanishes. But this possibility is excluded by condition (15). If

$$|n-n(k)| \leqslant \epsilon^{\frac{1}{2}}M_0,$$

${}_2b(k,n,T)$ is not defined.

For each k_1 we now *choose*

$$-2in(k_1)\,d\{{}_1a(k_1,T)\}/dT + 2\,{}_1a^*(k_2,T)\,{}_1a^*(k_3,T)\,e^{-iNT} = 0, \tag{20}$$

for each pair 2 and 3 of permitted normal modes (k_2, k_3 being regarded as the same pair as k_3, k_2) which satisfy

$$k_1+k_2+k_3 = 0,$$
$$-\epsilon^{\frac{1}{2}}M_0 \leqslant n_1+n_2+n_3 = \epsilon N \leqslant \epsilon^{\frac{1}{2}}M_0.$$

There will be relatively few permitted values of k_1 for which any such modes 2 and 3 exist, and provided
$$2\epsilon^{\frac{1}{2}}M_0 < |c_2-c_3|\,\delta k_0$$
there certainly will not be two such pairs. If there are none we set

$$d\{{}_1a(k_1,T)\}/dT = 0.$$

The effect of this choice of $d\,{}_1a/dT$ is that all the terms on the right-hand side of equation (18) except those giving rise to the particular integral of equation (19) cancel each other out. The general solution is thus

$${}_2\psi(x,t) = \sum_k {}_2a(k,T)\exp[i\{kx+n(k)\tau\}] + \sum_{k,n} {}_2b(k,n,T)\exp[i\{kx+n\tau\}]. \tag{21}$$

The first term is the general solution of the homogeneous equation, and describes oscillations in the normal modes of amplitude as yet unspecified. The bar on the summation sign of the second term indicates that the summation is over all distinct k and n defined by

$$k = k_2+k_3, \quad n = n(k_2)+n(k_3)$$

for permitted k_2 and k_3, but any terms which satisfy condition (15) have been omitted. Two pairs (k,n) are deemed identical only if they arise from k_2, k_3 and

k_3, k_2, respectively. This class is that of permitted forced oscillations of the second order.

Terms of order ϵ^3 give

$$\left\{\frac{\partial^2}{\partial\tau^2}+\frac{\partial^4}{\partial x^4}+\frac{\partial^2}{\partial x^2}+1\right\}{}_3\psi = \sum_k -\left\{\frac{d^2{}_1a}{dT^2}+2in\frac{d_2a}{dT}\right\}\exp\left[i\{kx+n(k)\tau\}\right]$$

$$+\sum_{k,n}-2in\frac{d_2b}{dT}\exp\left[i\{kx+n\tau\}\right]+\sum_{k_2,k_3,n_3}2\,{}_1a^*(k_2)\{{}_2a^*(k_3)+{}_2b^*(k_3,n_3)\}$$

$$\times\exp\left[-i\{(k_2+k_3)x+(n_2+n_3)\tau\}\right].$$

The pairs (k,n) describing the exponents are those for the normal modes in the first term on the right-hand side, for forced oscillations of the second order in the second term, and in the third term are

$$k = k_2+k_3, \quad n = n(k_2)+n(k_3)$$

for the first part, and in the second part

$$k = k_2+k_3, \quad n = n(k_2)+n_3,$$

where k_3, n_3 describes a permitted second order forced oscillation. We permit as forced oscillations of the third order any of these pairs which do not satisfy condition (15).

A particular integral for ${}_3\psi$ is

$$\sum_{k,n}{}_3b(k,n,T)\exp\left[i\{kx+n\tau\}\right],$$

where

$${}_3b(k,n) = \text{either } -2in\,(d_2b/dT),$$

$$\text{or} \quad \frac{2\,{}_1a(k_2)\,{}_1a(k_3)}{-(n_2+n_3)^2+\{n(k_2+k_3)\}^2},$$

$$\text{or} \quad \frac{2\,{}_1a(k_2)\,{}_2b(k_3,n_3)}{-(n_2+n_3)^2+\{n(k_2+k_3)\}^2}$$

according as how the permitted third order forced oscillation arises. The remaining terms on the right-hand side are cancelled if

$$-2in_1(d_2a_1/dT)+2({}_1a_2^*\,{}_2a_2^*+{}_1a_3^*\,{}_2a_2^*)\,e^{-iNT}$$

$$=(d^2/dT^2)\,{}_1a_1-\sum_{k_2',k_3',n_3'}2\,{}_1a_{2'}^*\,{}_2b_{3'}^*\,e^{-iN'T}. \tag{22}$$

Here, thanks to condition (17), there are at most two terms linear in ${}_2a$, arising from normal modes 2 and 3 which are partially resonant with mode 1, taken in either order. However, an infinite number of terms may enter on the right-hand side, but they consist of functions which have already been determined. A permissible term involves second order forced oscillations (k_3', n_3') where

$$k_3' = k_3''+k_4'', \quad n_3' = n_3''(k_3'')+n_4''(k_4'')$$

and hence indirectly primary oscillations of wave-number k_2', k_3'', k_4'', if

$$k_1+k_2'+k_3''+k_4'' = 0, \quad -\epsilon^{\frac{1}{2}}M_0 \leqslant n_1+n_2'+n_3''+n_4'' = \epsilon N' \leqslant \epsilon^{\frac{1}{2}}M_0.$$

These primary oscillations form a quartet which would partially resonate through cubic interactions. For any given $k_1, n(k_1)$ there may be an infinite number of these quartets, but only a finite number arising from primary oscillations in any given band of wave-numbers. Cubic interaction is an order of magnitude slower than quadratic resonance, and is taken into account here within a time T of order unity only as a secondary effect.

In a similar manner, from terms of order ϵ^4,

$$-2in,(d/dT)\,{}_3a_1+2\{{}_1a_2^*\,{}_2a_3^*+{}_1a_3^*\,{}_3a_2^*\}\,e^{-iNT}$$
$$=(d^2\,{}_2a_1/dT^2)-\sum_{\substack{k_2',n_2',\\k_3',n_3'}}2\{({}_2b_{2'}+{}_2a_{2'})\,({}_2b_{3'}+{}_2a_{3'})+{}_1a_{2'}\,{}_3b_{2'}\}\,e^{-iN'T}.$$

It is seen that the structure of this equation, and of the equations for natural oscillations of still higher order, is essentially the same as that of equation (22).

	L		N	L	R
${}_1a$	1	$\epsilon\,{}_1\psi$	ϵ	ϵ	ϵ^2
${}_2b,\,{}_2a$	$\epsilon^{-\frac{1}{2}}$	$\epsilon^2\,{}_2\psi$	ϵ^2	$\epsilon^{\frac{3}{2}}$	$\epsilon^{\frac{5}{2}}$
${}_3b,\,{}_3a$	ϵ^{-1}	$\epsilon^3\,{}_3\psi$	ϵ^3	ϵ^2	ϵ^3
${}_4b,\,{}_4a$	$\epsilon^{-\frac{3}{2}}$	$\epsilon^4\,{}_4\psi$	ϵ^4	$\epsilon^{\frac{5}{2}}$	$\epsilon^{\frac{7}{2}}$
...	...	...	...	...	...

TABLE 1. The nominal order (N), the least possible order (L) of various terms, and the least possible order of the remainder (R) after substitution into equation (1) of the approximation for ψ up to and including that nominal order.

A complete solution of equation (20) is given in the next section. Solutions of equation (22) are not available, though those of the homogeneous equation may presumably be obtained by differentiating those of equation (20) with respect to the initial conditions. However, because of the factors $e^{-iNT}, e^{-iN'T'}$, it is clear that for large $|N|$

$$O\left(\frac{d^2}{dT^2}\,{}_1a\right)\sim|N|\,O\left(\frac{d}{dT}\,{}_1a\right)$$

and

$$O\left(\frac{d^2}{dT^2}\,{}_2a\right)\sim|N|\,O\left(\frac{d}{dT}\,{}_2a\right),\quad\text{etc.}$$

But the maximum possible $|N|$ is $M_0\epsilon^{-\frac{1}{2}}$, and from equation (20) it is consistent to take both $d\,{}_1a/dT$ and ${}_1a$ of order unity, so the least possible order of the first term on the right-hand side of equation (22) is $O(\epsilon^{-\frac{1}{2}})$. From equation (19) if wave-numbers near where $n(k)=0$ are excluded, the least possible magnitude for ${}_2b$ is also $O(\epsilon^{-\frac{1}{2}})$.

With the assumption mentioned in § 4 that, as $|k|\to\infty$, ${}_1a(k)$ decreases sufficiently rapidly so that the sum of any series of terms formed from finite products of the ${}_1a(k)$ is convergent, with possible order no lower than that of some term in the series, we may now by inspection of equations (22), (14) and (21) assign a least possible order (L) to ${}_2b, {}_2a, {}_3b, {}_3a$, etc., and to successive terms in the expansion (12) which is applicable uniformly in the domain

$$|x|\leqslant X_0,\quad 0\leqslant\epsilon t=T\leqslant T_0$$

and then to the remainders ${}_1R, {}_2R$, etc., after substitution of these into equation (1). The sequence in the last column for the remainder is the same as equation (13) and justifies the expansion procedure.

6. Solutions for a resonant trio

Equation (20) for the rate of change of the amplitude function for the primary oscillation in one of the modes for a partially resonant trio is identical if $N = 0$ (exact resonance) with a scaled version of the first of equations (11), derived by a heuristic argument. Dropping the prefix on the amplitudes, we have

$$\begin{aligned} in_1(da_1/dT) &= a_2^* a_3^* e^{-iNT}, \\ in_2(da_2/dT) &= a_3^* a_1^* e^{-iNT}, \\ in_3(da_3/dT) &= a_1^* a_2^* e^{-iNT}, \end{aligned}$$

together with their complex conjugate forms, so

$$\frac{d}{dT}\{n_1 a_1 a_1^*\} = \frac{d}{dT}\{n_2 a_2 a_2^*\} = \frac{d}{dT}\{n_3 a_3 a_3^*\} = \mathscr{I}\{2a_1^* a_2^* a_3^* e^{-iNT}\}. \tag{23}$$

These equations have two independent integrals

$$n_1\{a_1 a_1^* - A_1^2\} = n_2\{a_2 a_2^* - A_2^2\} = n_3\{a_3 a_3^* - A_3^2\} = Z(T), \tag{24}$$

where A_1^2, A_2^2, A_3^2 are the initial values of the non-negative quantities $a_1(T)a_1^*(T)$, $a_2(T)a_2^*(T)$, $a_3(T)a_3^*(T)$, which measure the squares of the magnitudes of the primary oscillations without regard to phase. In order to satisfy the resonance condition (16), the frequencies n_1, n_2, n_3 cannot all have the same sign. Then equation (24) shows that if at any time T one of the magnitudes exceeds its initial values, at least one of the other magnitudes must be smaller than at $T = 0$. Equation (24) also shows that any change $Z(T)/n$ in the square of the magnitude of the oscillation in mode 1 is associated with changes $Z(T)/n_2$, $Z(T)/n_3$ in fixed proportion for modes 2 and 3. In any event

$$|Z(T)| \leqslant \max_{i=1,2,3} |n_i| A_i^2. \tag{25}$$

From the complex conjugates of equation (20) we also have

$$\begin{aligned} &(d/dT)\,\mathscr{R}\{2a_1^* a_2^* a_3^* e^{-iNT}\} \\ &\quad = \mathscr{R}\left\{ -\frac{2a_2 a_2^* a_3 a_3^*}{in_1} - \frac{2a_3 a_3^* a_1 a_1^*}{in_2} - \frac{2a_1 a_1^* a_2 a_2^*}{in_3} - iN2a_1^* a_2^* a_3^* e^{-iNT} \right\} \\ &\quad = N\mathscr{I}\{2a_1^* a_2^* a_3^* e^{-iNT}\} \\ &\quad = N(dZ/dT). \end{aligned}$$

So another integral of the equations is

$$\mathscr{R}\{2a_1^* a_2^* a_3^* e^{-iNT}\} = NZ(T) + H. \tag{26}$$

From the squares of equations (24) and (26)

$$\begin{aligned} &(dZ/dT)^2 + \{NZ(T) + H\}^2 = 4a_1 a_1^* a_2 a_2^* a_3 a_3^* \\ &\quad = 4/n_1 n_2 n_3 \{Z(T) + n_1 A_1^2\}\{Z(T) + n_2 A_2^2\}\{Z(T) + n_3 A_3^2\}. \end{aligned} \tag{27}$$

Equation (27) expresses the rate of change of the square of the magnitude in any mode as the square root of a cubic function of the cumulative change in the same quantity. It may be integrated explicitly in terms of elliptic functions involving in a complicated manner the constants A_1, A_2, A_3, H, determined by the initial conditions. However adequate qualitative information about the solutions may be obtained directly from the equations above. Equation (27) was first obtained by the author in collaboration with Dr H. K. Moffatt.

Values of $Z(T)$ which are solutions of equation (27) and of the initial conditions must be confined to a range between two roots of the cubic function

$$f(Z) \equiv 4/n_1 n_2 n_3 \{Z + n_1 A_1^2\}\{Z + n_2 A_2^2\}\{Z + n_3 A_3^2\} - \{NZ + H\}^2,$$

between which $f(Z)$ is positive.

For dZ/dT vanishes only when Z assumes a value Z' for which

$$f(Z') = 0.$$

If this is a simple zero $\quad df/dZ|_{Z'} \neq 0,$

and for $Z - Z'$ small,

$$d(Z - Z')/dT \sim \pm \sqrt{\{df/dZ|_{Z'}(Z - Z')\}},$$

so

$$Z - Z' \sim 4(df/dZ)|_{Z'}(T - T')^2$$

for some suitable constant T'. It thus appears that Z is restricted to one side only of the zero, and that the time taken for Z to move any small distance $|Z - Z'|$ into the zero and out again is strictly finite, unless $df/dz|_{Z'}$ vanishes.

$Z(T)$ cannot tend to infinity, because of the bounds set by equation (25). Thus it must oscillate between roots of $f(Z) = 0$ with a period which is finite. The only exception to this may occur if two of the three roots are equal and so df/dz vanishes there. In this case

$$Z - Z' \propto \exp\left[\pm \sqrt{\{\tfrac{1}{2}(d^2 f/dZ^2)|_{Z'}\}}\, T\right],$$

and the zero is not reached in a finite time. However, out of the set of possible initial values for A_1^2, A_2^2, A_3^2 and H this is a very exceptional case. We thus have the conclusion that every solution of equation (20) is a periodic function of T, with a period which depends on the initial conditions, but which is in general finite.

The crude upper bound given by equation (25) on the maximum transfer of amplitude squared into any one mode of a resonant trio may be refined if $|N|$ is large. Then the range of positive values of $f(Z)$ which includes $Z = 0$ is given approximately by

$$-(2A_1 A_2 A_3 + H)/N \leqslant Z(T) \leqslant (2A_1 A_2 A_3 - H)/N \quad \text{if} \quad |N| \gg 1.$$

The magnitudes of a_1, a_2, a_3 are thus almost constant, and it is easily seen that an asymptotic solution of equations (20) is

$$\left.\begin{aligned} a_1(T) &= a_1(0) + \frac{a_2^*(0)\,a_3^*(0)}{n_1 N}(e^{-iNT} - 1) + O\left(\frac{1}{N^2}\right), \\ a_2(T) &= a_2(0) + \frac{a_3^*(0)\,a_1^*(0)}{n_2 N}(e^{-iNT} - 1) + O\left(\frac{1}{N^2}\right), \\ a_3(T) &= a_3(0) + \frac{a_1^*(0)\,a_2^*(0)}{n_3 N}(e^{-iNT} - 1) + O\left(\frac{1}{N^2}\right). \end{aligned}\right\} \tag{28}$$

Each equation describes a primary oscillation of constant amplitude with a perturbation forced by the other two modes with relative frequency equal to the small departure from resonance of the trio, and amplitude inversely proportional to the same quantity. As $N \to \infty$, these perturbations match precisely with the forced oscillations described by ${}_2b(k,n)$ as $n - n(k) \to 0$. We are thus justified in saying that the interactions of primary oscillations due to quadratic resonances are significant only for trios satisfying the resonance conditions (16) with an error ϵN of order ϵ.

The quantity $Z(T)$ does not describe primarily a transfer of *energy* between the modes. The lowest order approximation for the exact energy density (equation (2)) is the same as for the linearized equation, and is partitioned between the three modes of a trio as

$$n_1^2 a_1 a_1^*, \quad n_2^2 a_2 a_2^*, \quad n_3^2 a_3 a_3^*,$$

whereas $Z(T)$ describes changes in

$$n_1 a_1 a_1^*, \quad n_2 a_2 a_2^*, \quad n_3 a_3 a_3^*,$$

However, equation (24) does not *contradict* conservation of energy, for to a lowest-order approximation,

$$\frac{d}{dT}\{n_1^2 a_1 a_1^* + n_2^2 a_2 a_2^* + n_3^2 a_3 a_3^*\} = (n_1 + n_2 + n_3)\frac{dZ}{dT} = 0.$$

7. Cubic resonances

In the event that there are no quadratic resonances possible at all, the solution of equation (20) for the primary oscillation is simply

$${}_1a(k,T) = \text{const.} = A(k).$$

Then
$${}_2b(k_3'' + k_4'', n_3'' + n_4'') = \frac{2A_{3''}A_{4''}}{-(n_3'' + n_4'')^2 + \{n(k_3'' + k_4'')\}^2}$$

and equation (22) becomes

$$2in_1 \frac{d}{dT}{}_2a_1 = 2\Sigma \frac{{}_1A_{2'}^* A_{3''}^* A_{4''}^* e^{-iN'T}}{\{n(k_3'' + k_4'')\}^2 - (n_3'' + n_4'')^2}, \tag{29}$$

where the summation is over all trios $2'$, $3''$, $4''$ which are partially resonant with mode 1,

$$\left.\begin{aligned} k_1 + k_2' + k_3'' + k_4'' &= 0, \\ -\epsilon^{\frac{1}{2}} M_0 \leqslant n_1 + n_2' + n_3'' + n_4'' &= \epsilon N^1 \leqslant \epsilon^{\frac{1}{2}} M_0. \end{aligned}\right\} \tag{30}$$

No terms $A_1^* {}_2a_2^* + A_2^* {}_2a_3^*$ enter on the left-hand side for there are no permitted pairs forming with 1 even a partially resonant trio. Equation (29) shows that if $N' = 0$, the amplitude in mode 1 will increase linearly, until after a time $T \sim \epsilon^{-1}$,

$$\epsilon^2 {}_2a \sim \epsilon {}_1a,$$

and the expansion procedure has broken down, because of cubic resonances.

Cubic resonances, in the absence of quadratic ones, may be described by redefining T as $\epsilon^2 t$, and proceeding essentially as before. The equations for the primary oscillations are

$$in_1 \frac{da_1}{dT} = \Sigma \frac{1}{\{n(k_3' + k_4')\}^2 - (n_3' + n_4')^2} a_{2'}^* a_{3'}^* a_{4'}^* e^{-iN'T}. \tag{31}$$

This is very similar to equation (29), except that a_2^*, etc., are now functions of T, and the derivative is of ${}_1a(k_1, T)$, not ${}_2a(k_1, T)$. However, unlike equation (20), it is not in general possible to dispense with the summation sign, for a type of term which always contributes is

$$k_2' = -k, \quad k_3' = -k_4',$$

which satisfies the resonance conditions identically. Any primary mode for which a_3' is not zero must be included here, although the effect is not to transfer energy directly into mode 1, but to introduce a phase shift which modifies the transfer from modes which satisfy the resonance condition in a less trivial manner.

To see this, suppose primary oscillations are present in only four modes, which satisfy the resonance condition (30). Then

$$\left.\begin{aligned}
in_1(da_1/dT) &= (c_{11}a_1a_1^* + c_{12}a_2a_2^* + c_{13}a_3a_3^* + c_{14}a_4a_4^*)\,a_1 + d_1a_2^*a_3^*a_4^*\,e^{-iNT},\\
in_2(da_2/dT) &= (c_{21}a_1a_1^* + c_{22}a_2a_2^* + c_{23}a_3a_3^* + c_{24}a_4a_4^*)\,a_2 + d_2a_3^*a_4^*a_1^*\,e^{-iNT},\\
in_3(da_3/dT) &= (c_{31}a_1a_1^* + c_{32}a_3a_3^* + c_{33}a_3a_3^* + c_{34}a_4a_4^*)\,a_3 + d_3a_4^*a_1^*a_2^*\,e^{-iNT},\\
in_4(da_4/dT) &= (c_{41}a_1a_1^* + c_{42}a_2a_2^* + c_{43}a_3a_3^* + c_{44}a_4a_4^*)\,a_4 + d_4a_1^*a_2^*a_3^*\,e^{-iNT}.
\end{aligned}\right\} \tag{32}$$

The constants $c_{ij}\,(ij = 1, 2, 3, 4)$ are all real but in general $c_{ij} \neq c_{ji}$ and

$$d_1 = \frac{1}{n^2(k_3+k_4)-(n_3+n_4)} + \frac{1}{n^2(k_4+k_2)-(n_4+n_2)^2} + \frac{1}{n^2(k_2+k_3)-(n_2+n_3)^2}.$$

The coefficient of a_1 in the first term is always real, and in the absence of the second interaction term on the right-hand side, it would make a_1 vary sinusoidally with T without change of magnitude.

Equations similar to equation (32) were obtained for ocean waves by Benney (1962), using the heuristic argument of § 3. For ocean waves the only permitted resonance conditions are of the form

$$\begin{aligned}
k_1 + k_2 &= k_3 + k_4,\\
|n_1| + |n_2| &= |n_3| + |n_4|.
\end{aligned}$$

These were slightly misstated by Benney, and according to Phillips (private communication) on reworking his analysis the coefficients $d_i\,(i = 1 \ldots 4)$ of the last terms (the interaction coefficients) all turn out to be equal. This was predicted by Hasselmann (1963) from considerations of conservation of energy and momentum. In this problem (in which only energy is conserved) it appears not to be so. The phase shift terms have also been studied for water waves by Longuet-Higgins & Phillips (1962). They include the increase of phase velocity for a single wave-train of finite amplitude. This arises from cubic interactions of the wave with itself, and thus do not figure in the quadratic resonance theory.

Although it is not possible in general, even for oscillations, to isolate individual resonant quartets for separate solution (because of the phase shift terms) it is worth noting that the method of solution of § 6 works also for equations (31), and the four linked non-linear equations with 20 disposable constants have a general analytical solution.

The integrals

$$Z(T) = \frac{n_1}{d_1}\{a_1 a_1^* - A_1^2\} = \frac{n_2}{d_2}\{a_2 a_2^* - A_2^2\} = \frac{n_3}{d_3}\{a_3 a_3^* - A_3^2\} = \frac{n_4}{d_4}\{a_4 a_4^* - A_4^2\} \tag{33}$$

were noted by Benney. The restrictions on $Z(T)$ are not so evident as in § 6, because it is not obvious that the n_i/d_i do not all have the same sign, but some bounds are implied by conservation of the total energy in the four modes. For ocean waves, when the d_i are all equal, these are the same as before.

Also
$$dZ/dT = \mathscr{I}\{2a_1^* a_2^* a_3^* a_4^* e^{-iNT}\} \tag{34}$$

and
$$d\,\mathscr{R}\{2a_1^* a_2^* a_3^* a_4^* e^{-iNT}\}/dT = \{N - (e_1 a_1 a_1^* + e_2 a_2 a_2^* + e_3 a_3 a_3^* + e_4 a_4 a_4^*)\}(dZ/dT)$$

where
$$e_j = \sum_i c_{ij}/n_i \quad (j = 1 \ldots 4).$$

Thus another integral of equation (21) is

$$\mathscr{R}\{2a_1^* a_2^* a_3^* a_4^* e^{-iNT}\} = (N - \sum_j e_j A_j^2) Z(T) - \tfrac{1}{2}(\sum_j (d_j/n_j) e_j) Z^2(T) + H. \tag{35}$$

From equations (33), (34) and (35),

$$\left(\frac{dA}{dT}\right)^2 = 4\left\{\frac{d_1}{n_1}Z + A_1^2\right\}\left\{\frac{d_2}{n_2}Z + A_2^2\right\}\left\{\frac{d_3}{n_3}Z + A_3^2\right\}\left\{\frac{d_4}{n_4}Z + A_4^2\right\} - \{H + (N - \sum_j e_j A_j^2) Z - \tfrac{1}{2}(\sum_j (d_j e_j/n_j) Z^2\}^2.$$

The right-hand side of this is a quartic polynomial in Z, with coefficients depending on the initial conditions. We may at once draw the conclusion that the general solution of equation (32) is a periodic function of T, with period depending on the initial conditions but which is only in exceptional cases infinite.

If oscillations in more than one resonant quartet are present simultaneously it is worth noting that the interactions between the quartets through the phase shift terms are of a rather specialized form. The integrals (33) exist for each quartet separately, but equation (35) for any particular quartet may now be obtained only by integration by parts, and will contain terms like $Z(dZ'/dT)$ where Z' is the function Z for another quartet. On squaring equations (34) and (35) the final result will be a set of quadratic simultaneous equations for the derivatives dZ/dT, dZ'/dT, etc., in terms of polynomials in Z, Z', etc. Numerical integration would probably be required but involving only *one* real parameter Z for each quartet—a considerable simplification of the original equations.

If, however, the quartets are so numerous that more than one set of modes $2'$, $3'$, $4'$ resonates with mode 1 in a non-trivial manner, then more than one interaction term (e.g. $d_1 a_2^* a_3^* a_4^* e^{-iNT}$) should appear in equation (32), and the solutions are linked in a much more general way. This occurs with resonant *waves* ($\delta k_0 = 0$), for which no explicit solutions of the resonance equations are known.

It is not altogether surprising that only 4 real integrals of the 4 complex equations (32) should be necessary to determine a solution when we recall that

only the *relative* phase $\arg\{a_1^* a_2^* a_3^* a_4^* e^{-iNT}\}$ can enter the interaction terms. For the complex amplitude function contains full information about the oscillation, about its phase as well as its amplitude. But the essence of resonance is that significant interactions only occur if the relative phase remains approximately constant over a large number of periods. After the equations have been simplified to pick out resonant interactions, the individual phases cannot enter them, even in slowly varying form.

8. Conclusions

The principal result of this paper is that the phenomenon of resonance may be put on a sound theoretical basis in terms of asymptotic expansions for small amplitude oscillations, at least for the model equation considered here. The primary oscillations interact by quadratic terms through independent resonant trios, with characteristic interaction time $\hat{t}$ inversely proportional to the amplitude ϵ of the oscillations, and the tolerance with which the resonance condition must be satisfied for significant interaction also varies as ϵ. If the resonance condition is not satisfied within this tolerance by any trio at all containing a given mode, the amplitude of the primary oscillation in that mode remains constant over a time t of order ϵ^{-1}, but may vary significantly if $t \sim \epsilon^{-2}$. For a partially resonant trio within the tolerance, the complex primary amplitudes in each mode are periodic functions of time with a period depending on the initial conditions but of order ϵ^{-1}. A quantity proportional to the square of the magnitude of the oscillations is transferred periodically between the modes of a trio, and energy is conserved during the transfer. If the resonance condition is only satisfied by the trio to a poor approximation, the transfer is small, and the solution could alternatively be described as a constant amplitude primary oscillation in each mode with a small amplitude secondary forced oscillation of nearly the same frequency forced by the other two modes.

The higher-order terms in the asymptotic expansion describe both forced oscillations of frequency substantially different from natural oscillations of the same wave-number, and natural oscillations of amplitude smaller than the primary ones. The transition between natural and forced oscillations is adequately covered in the theory. Although the theory is arranged as an expansion in powers of ϵ, the order of magnitude of the error made by truncating after successive terms increases only in powers of $\epsilon^{\frac{1}{2}}$. This estimate holds for any interval of $t \leqslant O(\epsilon^{-1})$, but the procedure definitely breaks down if $t \sim \epsilon^{-2}$. The number of primary oscillations which may influence any given mode increases rapidly with the nominal order of the oscillation in that mode, but is still effectively finite at each stage if the amplitude in the primary oscillations decreases rapidly enough with increasing wave-number. The influence is weak because the interaction is of high order and the amplitude small, and also because it has time to act limited to $O(\epsilon^{-1})$.

The methods used here and the general conclusions appear to be valid for any set of equations admitting quadratic resonances, provided that the linearized equation describes dispersive waves with a frequency relation independent of position which admits only one resonant trio containing any given mode, and

provided the group velocities appropriate to the three modes in the trio are all distinct. Values of the wave-number for which different branches of the frequency relation cross or coalesce, or for which the frequency vanishes, require special treatment. The fundamental assumption which is made throughout this paper is that attention is restricted to solutions which may be described as periodic functions of x, with period $4X_0$ which is large compared to a wavelength but strictly finite. This would be ensured by reflecting boundaries at $x = \pm X_0$. If ϵX_0 is not small the trios of resonant primary oscillations cannot be described as independent.

Primary oscillations for which no quadratic resonances are possible may still interact through cubic terms, but the characteristic time is of order ϵ^{-2}, and lies outside the scope of this expansion. However, if no quadratic resonances occur at all, the expansion may easily be rearranged, and it is seen that transfers of energy occur between modes forming resonant quartets. These are not independent, for the rate of transfer within a quartet may be modified through terms associated with the magnitude (but not the phase) of oscillations in modes outside the quartet. If only four primary oscillations are present, in modes forming a partially resonant quartet, the interaction equations, which are similar to those proposed for ocean waves by Benney (1962), may be solved for general initial conditions. The solutions are almost all periodic functions of time, with characteristic period $\hat{t} \sim \epsilon^2$. If more than four primary oscillations are present, a general solution is not available.

This analysis cannot be applied to the physically more significant case when no constraint of periodicity in x is imposed and a continuous spectrum of waves is present.

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